Diffusion of epicenter of earthquake aftershock, Omori's law, and generalized continuous-time random walk models [Helmstetter & Sornette, 2002b]

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In these series of paper, authors derive many of empirical laws of earthquake by ETAS model.

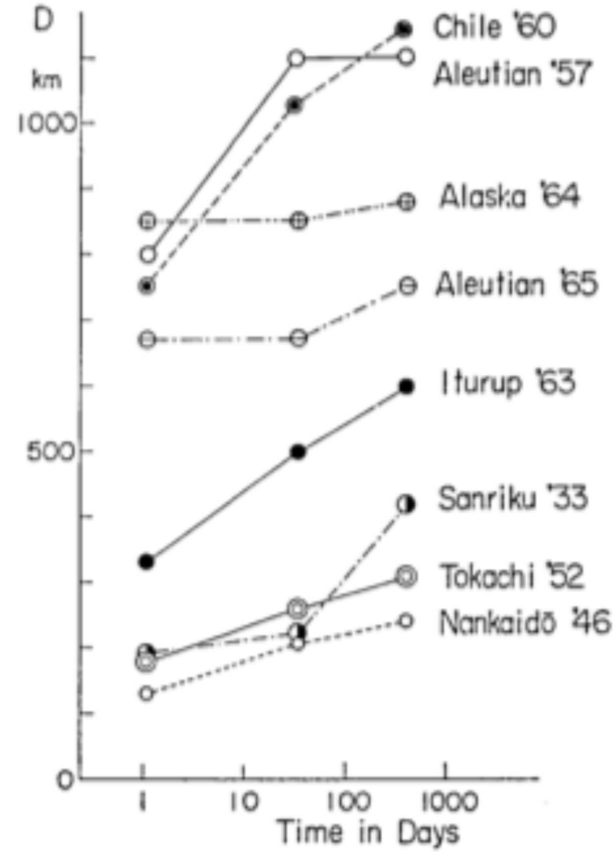
- Sornette and Sornette, 1999
- Helmstetter and Sornette, 2002a
- Sornette and Helmstetter, 2002

In this paper [Helmstetter and Sornette, 2002b], we investigate **aftershock diffusion**.

Aftershock diffusion

- from 1 km/h to 1 km/year
- Not universally observed

Why diffuse ?



Mogi, 1968

Why diffuse ?

・Viscous relaxation process (Rydelek and Sacks, 2001)

・Fluid transfer (Noir et al, 1997 , Nur and Booker, 1972, Hudnut et al, 1989)

・Rate and State friction's law and non-uniform stress (Dieterich, 1994)

・Cascade process : Large aftershocks reproduce their secondary aftershocks close to them. (this paper)

Flow

2. The ETAS model

Formulate ETAS model and refer the property of the model.
Numerical simulation.

3. Mapping of the ETAS model on the CTRW model

Derive the master equation of ETAS.
Establish a correspondence between the ETAS model and the CTRW (Continuous Time Random Walk model).

4. critical regime n=1

Derive the joint probability distribution N(t,r)
Calculate the average distance between mainshock and its aftershock R as a power law function of elapsed time. (R~t^H)

5. New Question on Aftershocks derived from the CTRW Analogy

6. Discussion

Summarize result of different regime
Comparison to related study

7. Conclusion

2. The ETAS model

Formulate ETAS model and refer the property of the model.
Numerical simulation.

ETAS Model

$$\boxed{\phi_{m_i}(t-t_i, \vec{r}-\vec{r}_i) = \rho(m_i)\Psi(t-t_i)\Phi(\vec{r}-\vec{r}_i).}$$

m_i : magnitude r_i : positon t_i : time

‘bare propagator’ = seismic rate **directly** induced by a single ‘mother’ *i*

(1) Large earthquake reproduce many aftershocks.

$$\rho(m_i) = K10^{\alpha(m_i - m_0)},$$

(2) Normalized waiting time distribution = ‘bare’ omori’s law

$$\Psi(t) = \frac{\theta c^{\theta}}{(t+c)^{1+\theta}} H(t),$$

$\theta > 0, H(t)$ is Heaviside function

(3) Normalized spatial ‘jump’ distribution = isotropic elastic Green function dependence

$$\Phi(\vec{r}) = \frac{\mu}{d\left(\frac{|\vec{r}|}{d}+1\right)^{1+\mu}},$$

$\mu > 0$

α and b

event-size distribution = GR law

$$P(m) = b\ln(10)10^{-b(m-m_0)}, \quad (6)$$

number of daughter

$$\rho(m_i) = K10^{\alpha(m_i-m_0)}, \quad (3)$$

$\alpha > b$: large event dominate earthquake triggering
$\alpha < b$: small event dominate earthquake triggering

recent reanalysis of seismic catalogs indicates $\alpha < b$ and α =0.8 (Helmstetter, 2003)

but case of α >0.5 is difficult to analyze (infinite variance $\rho(m)$)

therefore our model uses $\underline{b = 1, \alpha = 0.5}$

branching ratio *n* (Helmstetter & Sornette, 2002a)

n : *average* # of daughter created per mother event (summed by all possible magnitude)

$$n \equiv \int d\vec{r} \int_{t_i}^{+\infty} dt \int_{m_0}^{+\infty} dm_i P(m_i) \phi_{m_i}(t-t_i, \vec{r}-\vec{r}_i)$$

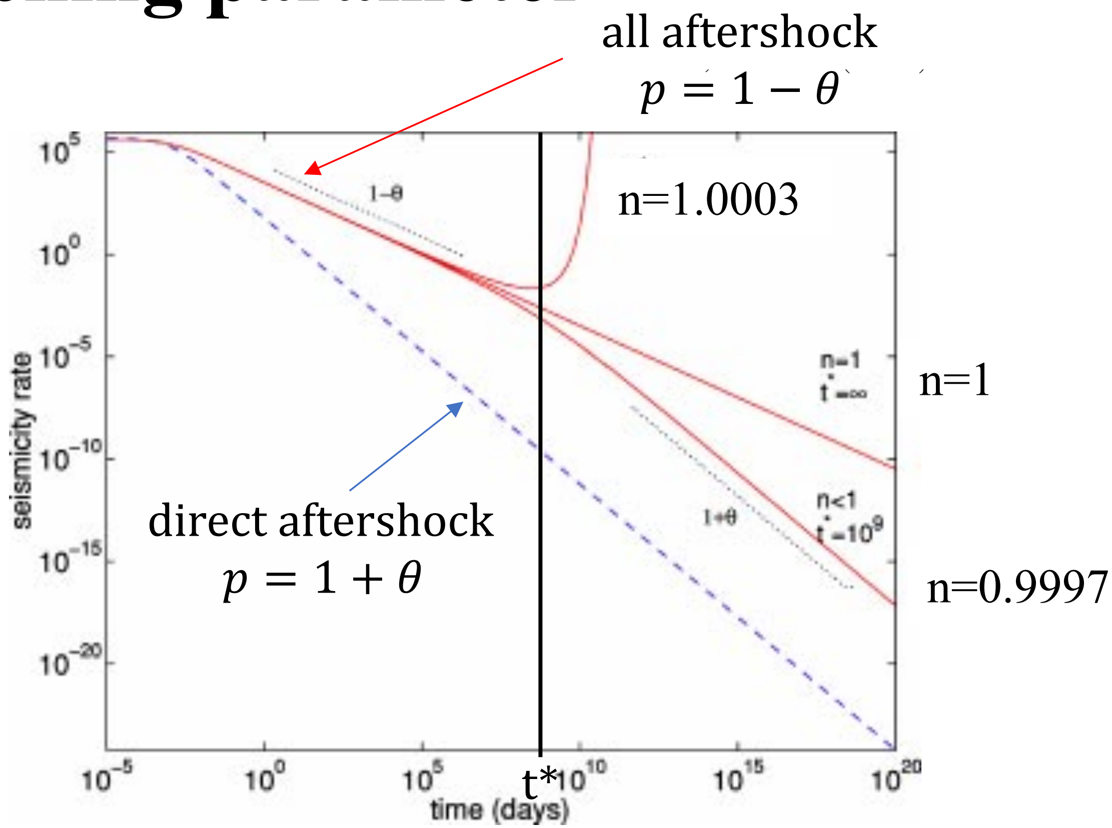
$$= \int_{m_0}^{+\infty} dm_i P(m_i) \rho(m_i) = \frac{Kb}{b-\alpha},$$

due to cascades of aftershocks, total # of event is larger by the factor 1/(1-n) ~ 10

→ n is a branching parameter

$n < 1$: subcritical regime (finally die out)
$n > 1$: supercritical regime (exponentially increase)
$n = 1$: critical regime (border between birth and death)

n is branching parameter



characteristic time

$$t^* = c\left(\frac{n\Gamma(1-\theta)}{|1-n|}\right)^{1/\theta}, \quad (1)$$

t<t*, all regime behave identically

Numerical simulation : method (Ogata, 1998 & 1999)

- **initial condition**
  t=0 r=0 M7 event occur

- **algorithm**
  decide time of next event by nonstationary poisson process (8)

  $$\lambda(t)=\sum_{t_i\leq t} K10^{\alpha(m_i-m_0)}\frac{\theta c^{\theta}}{(t-t_i+c)^{1+\theta}},$$

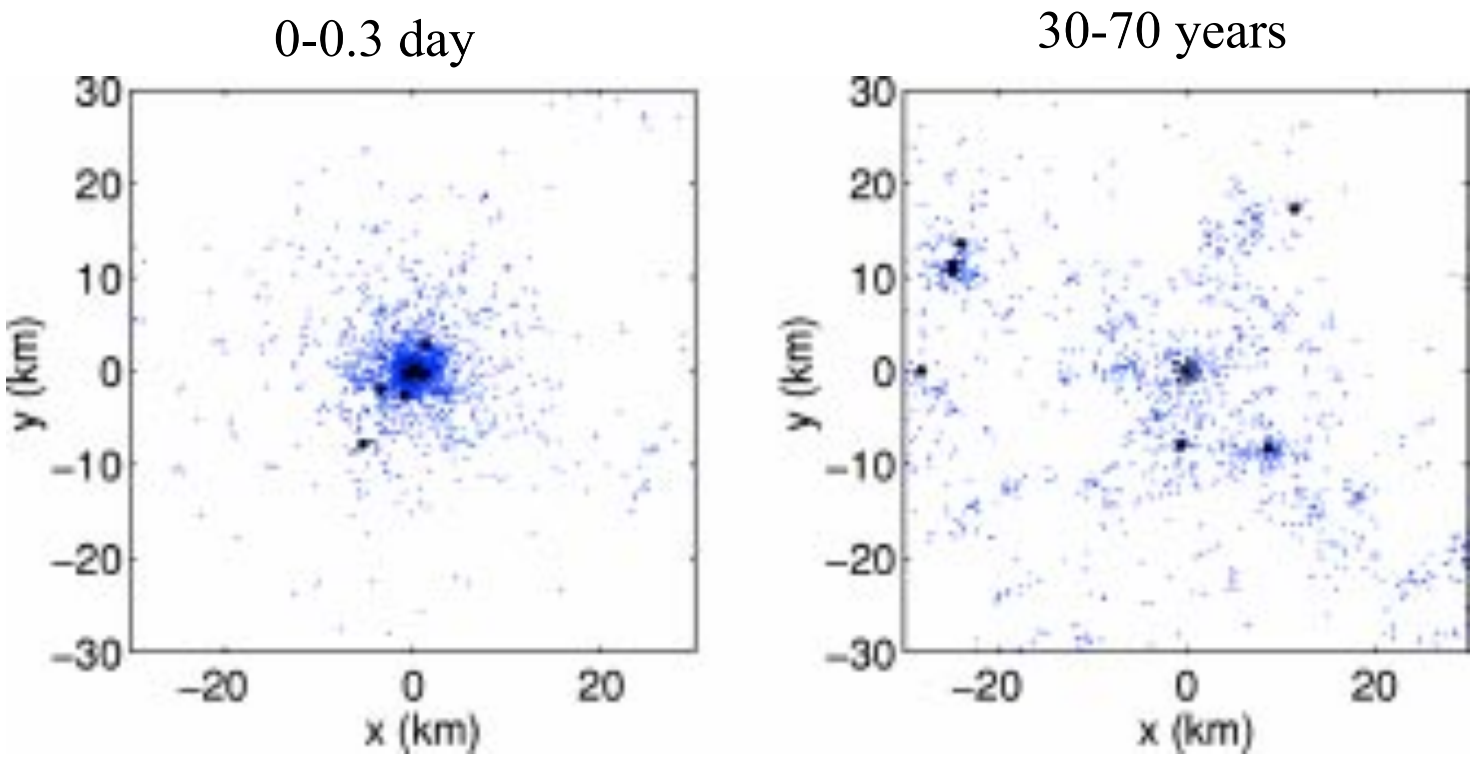
  → decide magnitude by GR law

  → select mother in all preceding events by (2) $\phi_{m_i}(t-t_i,\vec{r}-\vec{r}_i)=\rho(m_i)\Psi(t-t_i)\Phi(\vec{r}-\vec{r}_i).$

  → decide location of new event by (5) $\Phi(\vec{r})=\dfrac{\mu}{d\left(\dfrac{|\vec{r}|}{d}+1\right)^{1+\mu}},$

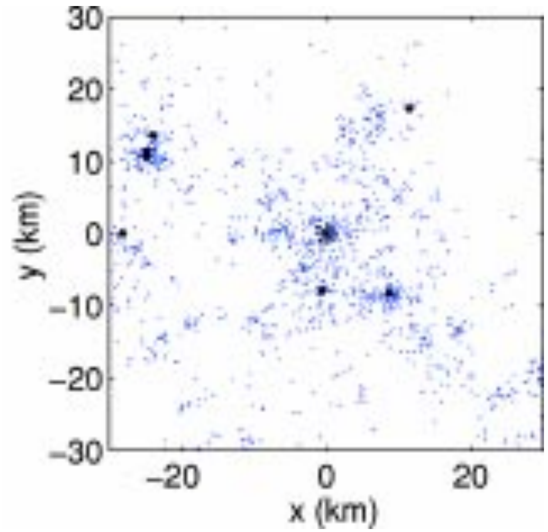
- **parameter set**
  $\theta=0.2, b=1, \alpha=0.5, n=1, \mu=1, m_0=0, d=1\text{km}, c=0.001\text{day}$

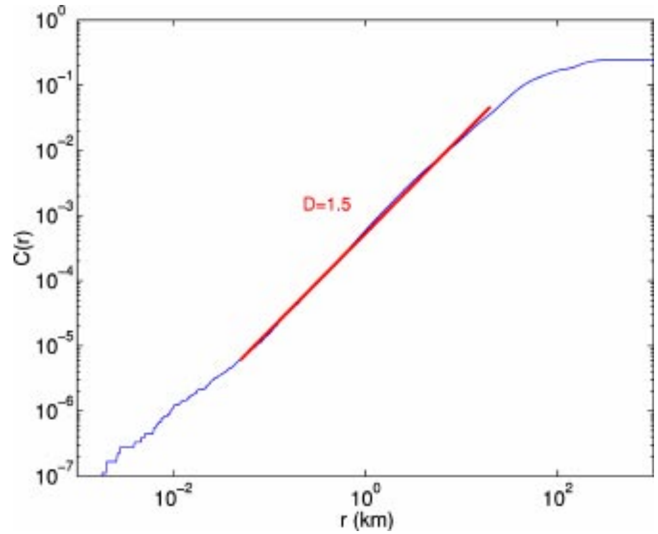
Numerical simulation : Result



considerable diffusion occurs

[30,70] yrs : fractal distribution





correlation dimension D ~1.5

[0,70]yrs : D ~1.85 [7,70]yrs : D ~1.7

reported active fault system: D = [1.65:1.95]

3. Mapping of the ETAS model on the CTRW model

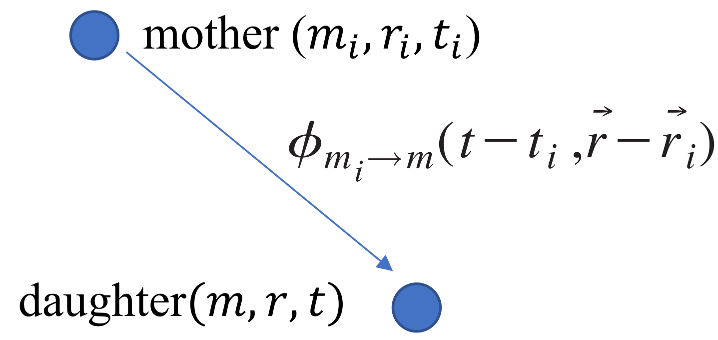
Derive the master equation of ETAS.
Establish a correspondence between the ETAS model and the CTRW (Continuous Time Random Walk model).

From direct Omori's law To renormalized Omori's law

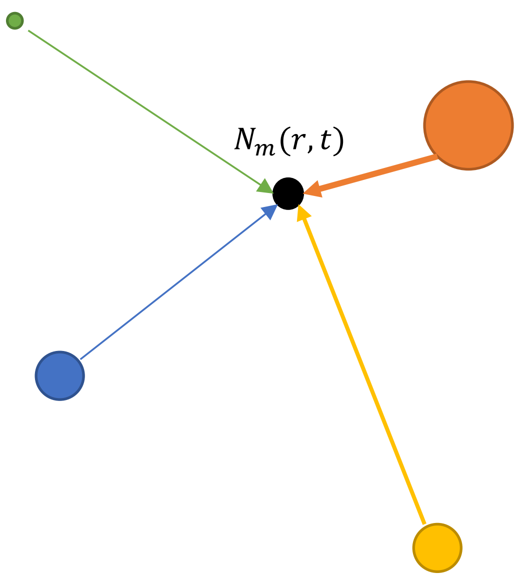
direct Omori law

$$\phi_{m_i \to m}(t-t_i, \vec{r}-\vec{r}_i) = \rho(m_i \to m)\Psi(t-t_i)\Phi(\vec{r}-\vec{r}_i),$$

$$\rho(m_i \to m) = n\ln(10)(b-\alpha)10^{\alpha(m_i-m_0)}10^{-b(m-m_0)}.$$



renormalized Omori law



source term (mainshock must occur at t = 0)

$$S(t,\vec{r},m) = \delta(t)\,\delta(m-M)\,\delta(\vec{r}),$$

$$N_m(t,\vec{r}) = S(t,\vec{r},m) + \int d\vec{r}' \int_{m_0}^{\infty} dm' \times \int_0^t d\tau\, \phi_{m' \to m}(t-\tau, \vec{r}-\vec{r}')N_{m'}(\tau,\vec{r}'). \quad (17)$$

# of event by cascade process

convolution

$$N_m(t,\vec{r}) = S(t,\vec{r},m) + \int d\vec{r}' \int_{m_0}^{\infty} dm' \times \int_0^t d\tau \phi_{m' \to m}(t-\tau, \vec{r}-\vec{r}') N_{m'}(\tau, \vec{r}'). \quad (17)$$

assumption : daughter's magnitude is independent of its mother

(GR preserved all time. It is adequate *only if* $\alpha \leq b/2$)

$N_m(t,r) = P(m)N(t,r)$ for $t > 0$

magnitude m vanishes

$$N(t,\vec{r}) = S_M(t,\vec{r}) + \int d\vec{r}' \int_0^t d\tau \phi(t-\tau, \vec{r}-\vec{r}') N(\tau, \vec{r}'),$$

$$t > 0, \quad (18)$$

$$S_M(t,\vec{r}) = \delta(r)\delta(t)\rho(M)/n,$$

Master Equation of ETAS = renormalized Omori's law

$N(t,r) = E[\lambda(t)\Phi(r)]$: Expectation value(1st moment)

Continuous time random walk model (Montroll & Weiss, 1965)

- generalization of naïve Random Walk model
  continuous distribution $\phi(r,t)$ of spatial step (jump length) and time step (wating time)
- master equation of CTRW is identical to ETAS

$$N(t,\vec{r}) = S_M(t,\vec{r}) + \int d\vec{r}\,' \int_0^t d\tau \phi(t-\tau, \vec{r}-\vec{r}\,') N(\tau, \vec{r}\,'),$$

$$t > 0, \qquad (18)$$

A) $\mathrm{N}(\mathrm{t},\mathrm{r})$: PDF for the random walker to ***Just*** arrive at r at t.
B) $\mathrm{S_M}(\mathrm{t},\mathrm{r})$: initial condition of random walk,
C) integral on (18) denote superposition of all possible paths just having arrived at r at t, weighted by a transfer function ϕ

- Therefore we can borrow the deep knowledge of CTRW for the understanding Earthquake clustering.

- N and W

  $N(t,r)$: PDF of just arriving at position r at time t

  $W(t,r)$: PDF of being at position r at time t

$$N(t,\vec{r}) = S_M(t,\vec{r}) + \int d\vec{r}\,' \int_0^t d\tau \phi(t-\tau, \vec{r}-\vec{r}\,') N(\tau,\vec{r}\,'), \quad (18) \qquad W(t,\vec{r}) = \int_0^t dt' \left[1 - \int_0^{t-t'} dt'' \Psi(t'') \right] N(t',\vec{r}). \quad (19)$$

- using Laplace-Fourier transform

$$\hat{N}(\beta,\vec{k}) = \frac{\hat{S}_M(\beta,\vec{k})}{1 - n\hat{\Psi}(\beta)\hat{\Phi}(\vec{k})}, \quad (21) \qquad \hat{W}(\beta,\vec{k}) = \frac{1-\hat{\Psi}(\beta)}{\beta}\hat{N}(\beta,\vec{k}). \quad (20)$$

- CTRW models *transport phenomena* in heterogeneous media. considering earthquake as *transport of stress* in heterogeneous crust, correspondence between ETAS and CTRW is natural ?

summary : correspondence between ETAS and CTRW

TABLE I. Correspondence between the ETAS (epidemic-type aftershock sequence) and CTRW (continuous-time random walk) models. "PDF" stands for probability density function.

	ETAS	CTRW
$\Psi(t)$	PDF for a "daughter" to be born at time t from the mother that was born at time 0	PDF of waiting times
$\Phi(\vec{r})$	PDF for a daughter to be triggered at a distance $\vec{r}$ from its mother	PDF of jump sizes
m	Earthquake magnitude	Tag associated with each jump
$\rho(m)$	Number of daughters per mother of magnitude m	Local branching ratio
n	Average number of daughters created per mother summed over all possible magnitudes	Control parameter of the random walk survival (branching ratio)
$n<1$	Subcritical aftershock regime	Subcritical "birth and death"
$n=1$	Critical aftershock regime	The standard CTRW
$n>1$	Supercritical exponentially growing regime	Explosive regime of the "birth and death" CTRW
$N(t,\vec{r})$	Number of events of any possible magnitude at $\vec{r}$ at time t	PDF of just having arrived at $\vec{r}$ at time t
$W(t,\vec{r})$	PDF that an event at $\vec{r}$ has occurred at a time $t' \leq t$ and that no event occurred anywhere from t' to t	PDF of being at $\vec{r}$ at time t

4. critical regime $n = 1$

Derive the joint probability distribution $N(t,r)$

Calculate the average distance between mainshock and its aftershock R as a power law function of elapsed time. ($R \sim t^H$)

space : Fourier transform

$$\Phi(r) = \frac{\mu}{d(r/d+1)^{1+\mu}} \quad (5)$$

- for $\mu > 2$, $\langle r^2 \rangle = \sigma^2$ (finite)

$$\hat{\Phi}(k) = 1 - \sigma^2 k^2 + O(k^o) \text{with } o > 2 \quad (23)$$

- for $0 < \mu \leq 2$, $\langle r^2 \rangle =$ infinite (so-called Levy-flight)

$$\hat{\Phi}(k) = 1 - \sigma^\mu k^\mu + O(k^o) \text{with } o > \mu \quad (24)$$

$$\sigma = \begin{cases} d[\Gamma(1-\mu)]^{1/\mu}, & 0<\mu<1 \\ \dfrac{d\pi}{\mu\Gamma(\mu-1)\sin(\pi\mu/2)}, & 1<\mu<2. \end{cases} \quad (25)$$

time : Laplace transform

$$\Psi(t) = \frac{\theta c^{\theta}}{(t+c)^{1+\theta}} H(t), \qquad (4)$$

for $\theta < 1$,

$$\hat{\Psi}(\beta) = 1 - (\beta c')^{\theta} + \mathcal{O}(\beta^{\omega}) \quad \text{with} \quad \omega \geqslant 1, \qquad (26)$$

$$c' = c\left(\Gamma(1-\theta)\right)^{\frac{1}{\theta}}$$

for small β and k,

$$\hat{N}(\beta,\vec{k}) = \frac{\hat{S}_M(\beta,\vec{k})}{1-n\hat{\Psi}(\beta)\hat{\Phi}(\vec{k})}, \quad (21) \longrightarrow \quad \hat{N}(\beta,\vec{k}) = \frac{\hat{S}_M(\beta,\vec{k})}{1-n+n(\beta c')^{\theta}+n\sigma^{\mu}k^{\mu}}. \quad (27)$$

- case n=1

$$\hat{N}(\beta,\vec{k}) = \hat{S}_M(\beta,\vec{k})\frac{1}{(\beta c')^{\theta}+(\sigma k)^{\mu}}. \quad (51)$$

$\longrightarrow$ Analyzed in detail below

- case n<1

$$\hat{N}(\beta,\vec{k}) = \frac{\hat{S}_M(\beta,\vec{k})}{1-n}\frac{1}{1+(\beta t^*)^{\theta}+(kr^*)^{\mu}}, \quad (29) \quad r^* = \sigma\left(\frac{n}{1-n}\right)^{1/\mu}. \quad t^* = c\left(\frac{n\Gamma(1-\theta)}{|1-n|}\right)^{1/\theta},$$

$t < t^*$ and $r < r^*$ $\longrightarrow$ Same expression as for n=1

otherwise $\longrightarrow$

$$\hat{N}(\beta,\vec{k}) \simeq \frac{\hat{S}_M(\beta,\vec{k})}{1-n}\frac{1}{1+(\beta t^*)^{\theta}}\frac{1}{(1+(kr^*)^{\mu}}. \quad (31)$$

N can be factorized : No diffusion

$\boldsymbol{\theta > 1, \mu > 2}$

$\widehat{\Phi}(k) = 1 - \sigma^2 k^2 + O(k^o)$with $o > 2$, $\hat{\Psi}(\beta) = 1 - (\beta c')^{\theta} + \mathcal{O}(\beta^{\omega}) \quad \text{with} \quad \omega \geqslant 1$,

$$\hat{N}(\beta, \vec{k}) = \frac{\hat{S}_M(\beta, \vec{k})}{1 - n\hat{\Psi}(\beta)\hat{\Phi}(\vec{k})}, \longrightarrow N(\beta, k) = S_M(\beta, k)\frac{1}{\beta c' + \sigma^2 k^2}$$

in real domain

$$N(t, \vec{r}) \propto \frac{1}{(Dt)^{d/2}} \exp[-(\vec{r})^2/Dt] \quad \text{where} \quad D = \sigma^2/c', \tag{33}$$

$R = \langle|\vec{r}|^2\rangle^{1/2} \sim t^H$ with H=0.5 : standard diffusion

But $\theta > 1$ is not appropriate
case of $\theta < 1$?

$\boldsymbol{\theta < 1, \mu > 2}$

From complicated calculation,

$$z = \frac{Dt^{\theta/2}}{|\vec{r}|} \quad (36)$$

for small z ($r \gg Dt^{\theta}/2$)

$$N(t,\vec{r}) = \frac{c'^{-\theta}}{2Dt^{1-(\theta/2)}} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\Gamma[(1-k)\theta/2]}. \quad (40)$$

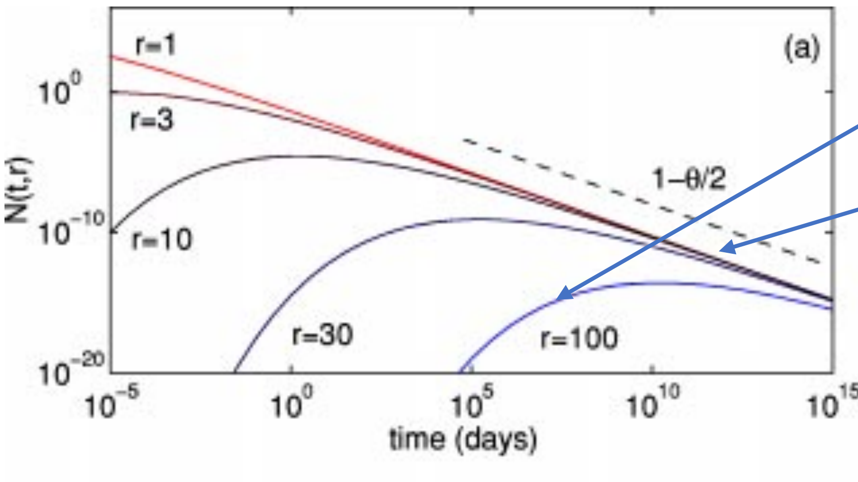
for large z ($r \ll Dt^{\theta}/2$)

$$N(t,r) \sim \frac{c'^{-\theta}}{Dt^{1-(\theta/2)}} \left(\frac{|\vec{r}|}{Dt^{\theta/2}} \right)^{(1-\theta)/(2-\theta)} \times \exp\left[-\left(1-\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)^{\theta/(2-\theta)} \left(\frac{|\vec{r}|}{Dt^{\theta/2}} \right)^{2/(2-\theta)} \right]. \quad (42)$$

N(t,r) cannot be factorized = **diffusion**

$R \sim t^{H}$ with H= $\theta/2$: subdiffusion

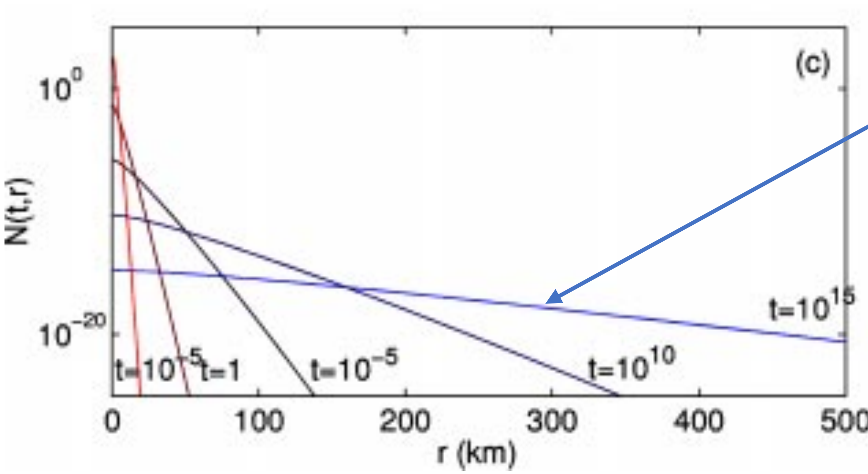
$\theta < 1, \mu > 2$



$t < \left(\frac{r}{D}\right)^{\frac{2}{\theta}}$: increase

$t > \left(\frac{r}{D}\right)^{\frac{2}{\theta}}$: power law decay (p =1-$\theta/2$)

but *global* decay exponent p= 1-θ

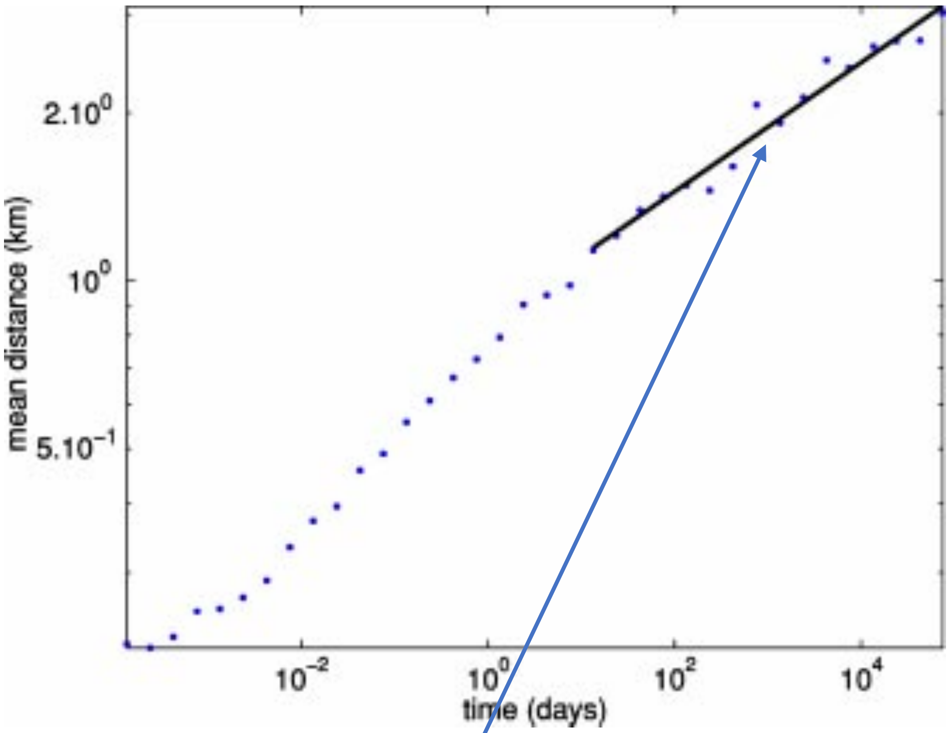


$N(t,r) \sim \exp[C(t)|\vec{r}|^q]$

$q = 2/(2-\theta) \sim 1$: exponential decay

C(t) define diffusion with time

Numerical simulation $\theta = 0.2, \mu = 3$



R ~t^H with H=0.12 (predicted H is 0.1)

$\boldsymbol{\theta < 1, \mu \leq 2}$

$$\hat{N}(\beta,\vec{k})=\hat{S}_M(\beta,\vec{k})\frac{1}{(\beta c')^{\theta}+(\sigma k)^{\mu}}. \quad \hat{W}(\beta,\vec{k})=\hat{S}_M(\beta,\vec{k})\frac{(\beta)^{\theta-1}c'^{\theta}}{(\beta c')^{\theta}+(\sigma k)^{\mu}}.$$

$R \sim t^H$ with $\mathrm{H} = \frac{\theta}{\mu}$: superdiffusion or subdiffusion

z expansion for small z and 1/z expansion for lagre z, $$z=\frac{Dt^{\theta/2}}{|\vec{r}|} \qquad (36)$$

for small z ($r \gg Dt^{\theta}/2$) $$N(t,\vec{r})=\frac{\sin\left(\frac{\pi\mu}{2}\right)}{\sigma c'\pi}\frac{\Gamma(1+\mu)}{\Gamma(2\theta)}\left(\frac{c'}{t}\right)^{1-2\theta}\left(\frac{\sigma}{|\vec{r}|}\right)^{1+\mu} \qquad (59)$$

$p = 1 - 2\theta$

for large z ($r \ll Dt^{\theta}/2$) $$N(t,\vec{r})=\frac{c^{-\theta}}{D\pi\mu t^{1-\theta+\theta/\mu}}\sum_{m=0}^{+\infty}(-1)^m$$

$$\times\left[\mu z^{1-\mu-m\mu}\frac{\Gamma(1-(m+1)\mu)\sin((m+1)\mu\pi/2)}{\Gamma(-m\theta)}\right.$$

$$\left.+\frac{z^{-m}}{m!}\frac{\pi\cos(m\pi/2)}{\sin[(m+1)\pi/\mu]\Gamma(\theta-(m+1)\theta/\mu)}\right]. \qquad (61)$$

N(t,r) for large z can be further classified

$$N(t,r) \simeq \frac{\Gamma(1-2\mu)\sin(\pi\mu)\sin(\pi\theta)}{c'\sigma\pi^2} \frac{\Gamma(1+\theta)}{(r/\sigma)^{1-2\mu}} \frac{1}{(t/c')^{1+\theta}}$$

for $\mu < 0.5,$

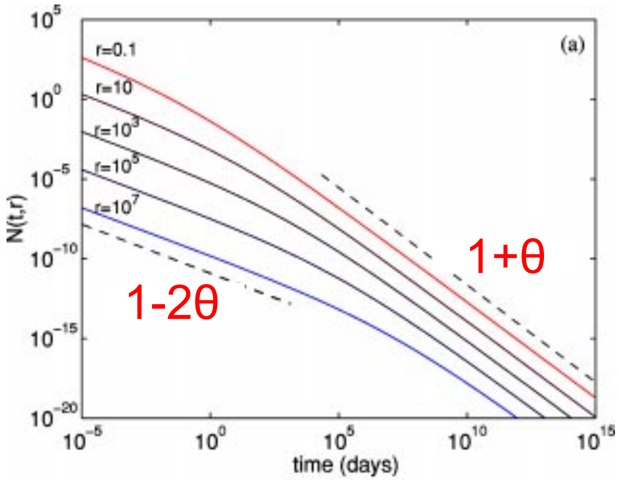
$$\boxed{p = 1 + \theta}$$

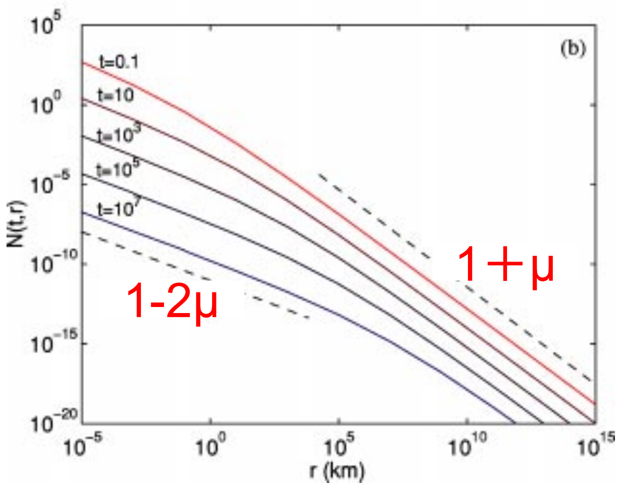
$$N(t,r) \simeq \frac{c'^{-\theta}}{c'\sigma\mu\Gamma(\theta-\theta/\mu)\sin(\pi/\mu)} \frac{1}{(t/c')^{1-\theta+\theta/\mu}}$$

for $0.5 < \mu < 2.$ (62)

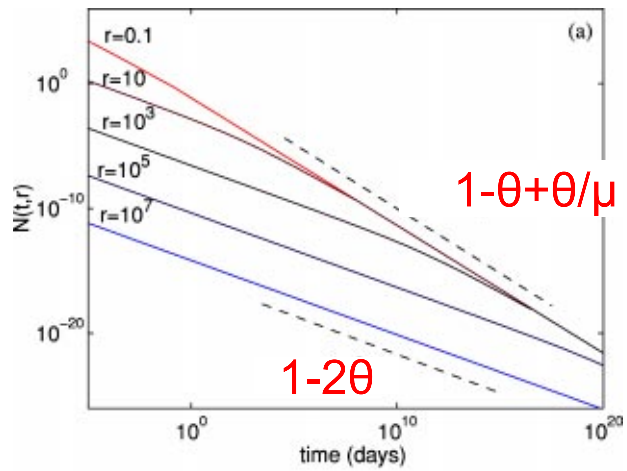
$$\boxed{p = 1 - \theta + \frac{\theta}{\mu}}$$

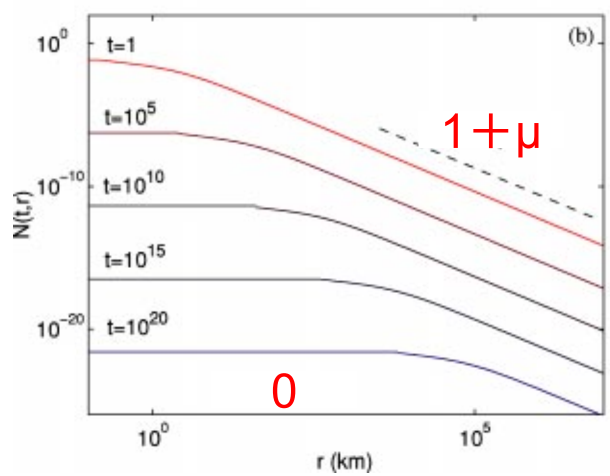
θ=0.2, μ=0.2



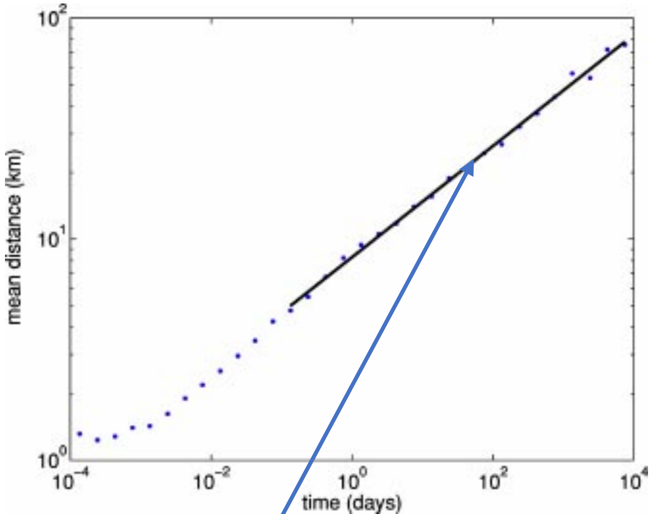


θ=0.2, μ=0.9

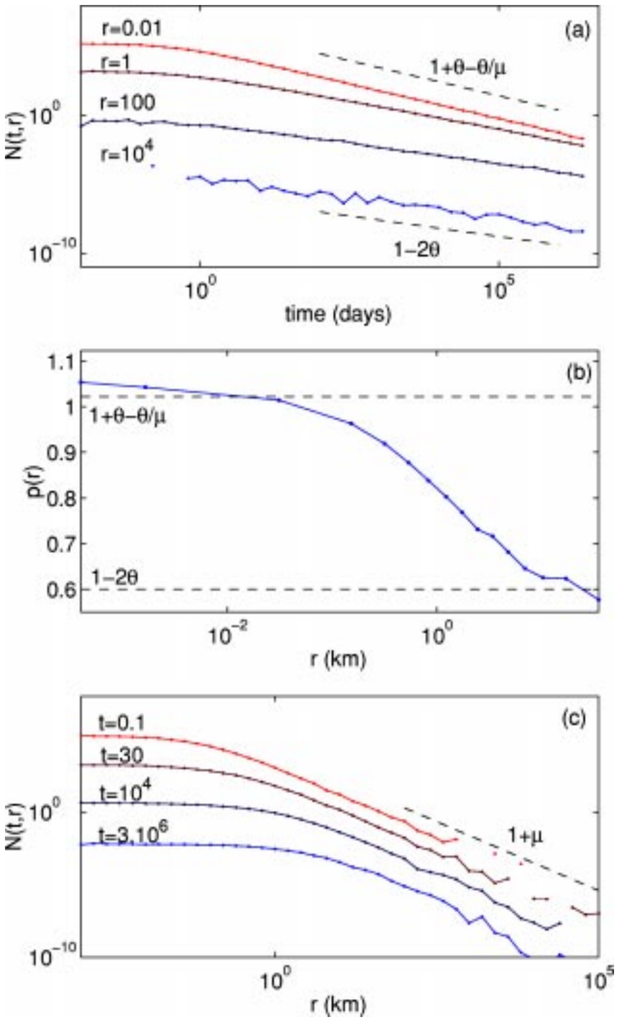




Numerical simulation θ=0.2, μ=0.9



R ~t^H with H=0.25 (predicted H is 0.22)



averaging over 500 sample

Other distribution $\Psi(t)$ and $\Phi(r)$

- $\Psi(t)$ and $\Phi(r)$ are not power law

$$\Psi(t) = \lambda e^{-\lambda t} \quad \Phi(\vec{r}) = L_{\mu}(|\vec{r}|)$$

$\longrightarrow$ $R \sim t^{H}$ with $\mathrm{H} = \frac{1}{\mu}$: superdiffusion

at large time $r \ll (\lambda t)^{\frac{1}{\mu}}, N(t,r) \sim 1/t^{\frac{1}{\mu}}$

Despite $\Psi(t)$is exponential distribution, local Omori's law $p = 1/\mu$ is generated

constant seismic rate for n=1

- nonseparable bare propagetor = microscopic diffusion process embodied

$$\phi_{m_i}(t-t_i, \vec{r}-\vec{r}_i) = \rho(m_i)\Psi(t-t_i)\Phi(|\vec{r}-\vec{r}_i|/\sqrt{Dt}),$$

$$\Phi(|\vec{r}-\vec{r}_i|/\sqrt{Dt}) = \frac{1}{\sqrt{2Dt}}\exp(-|\vec{r}-\vec{r}_i|^2/Dt).$$

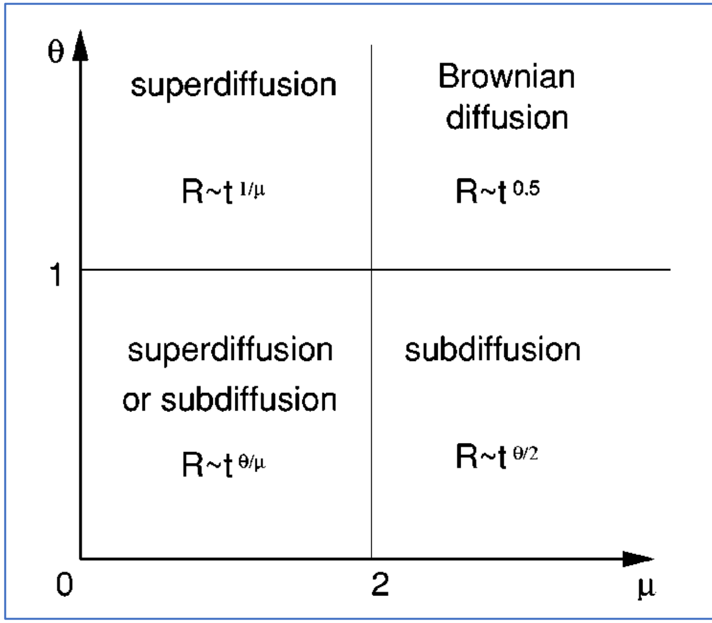
$$\longrightarrow \quad N(t,\vec{r}) \sim \frac{1}{t^{1-\theta}} \frac{1}{\sqrt{2\pi Dt}}\exp(-|\vec{r}|^2/Dt).$$

$R \sim t^{H}$ with $\mathrm{H} = 0.5$: standard diffusion

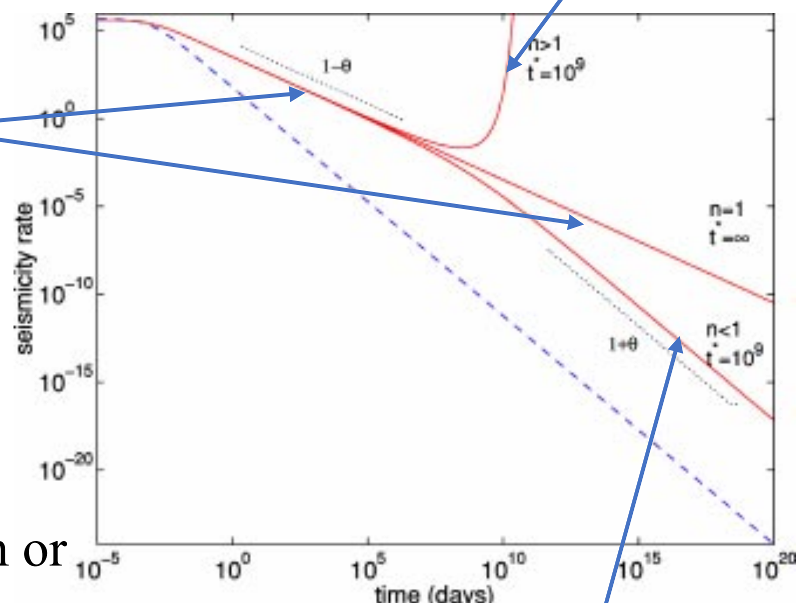
6. Discussion

Summarize result of different regime
Comparison to related study

Diffusion exponent



rapid diffusion



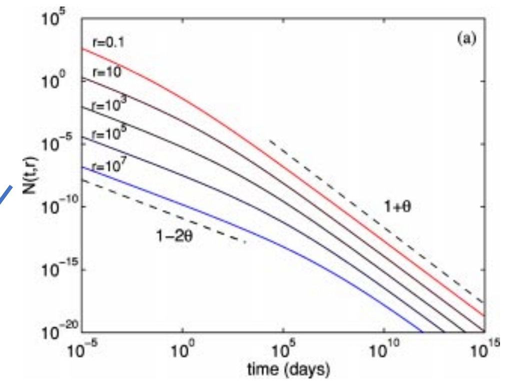
no diffusion

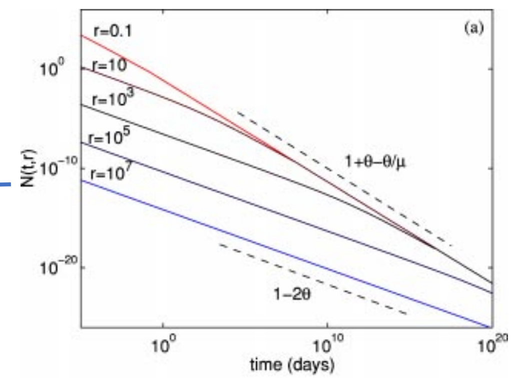
ETAS predicts that seismic diffusion or subdiffusion occurs and should be observable only when the observed Omori exponent is less than 1.

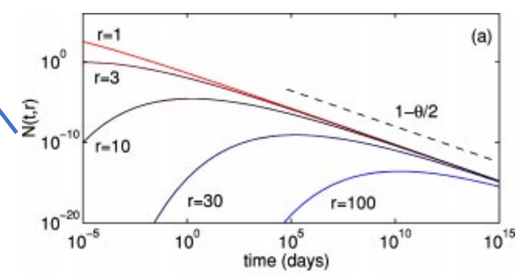
however, it is difficult to test on seismicity data.

Omori exponent (n = 1)

	Large z $(r \ll Dt^H)$	Small z $(r \gg Dt^H)$
$\mu < 0.5$	$p = 1 + \theta$	$p = 1 - 2\theta$
$0.5 \leqslant \mu < 2$	$p = 1 - \theta + \theta/\mu$	$p = 1 - 2\theta$
$2 \leqslant \mu$	$p = 1 - \theta/2$	Not defined [a]







Comparison to related research

- **Noir et al., 1997** (1989 DobiEQ sequence)
  H =0.5 due to fluid transfer
- **Tajima & Kanamori, 1985** (subduction zone)
  logarithmic or H=0.1diffusion
- **Shaw, 1993** (California)
  no diffusion and p~1 ← $\theta \sim 0$, very small H ?
- **Dieterich, 1994** (RSF law)
  aftershock zone expand but not grow as power law.
- **Marsan et al, 2000** (several catalogs)
  H=0.2 ← apparent diffusion due to their analysis method (counting uncorrelated events)
- **Sotolongo-Costa et al., 2000** (microearthquakes in Spain)
  interpreted sequence of earthquakes as a random walk process
  ↑ different from this paper (identify sequence as a *single* CTRW)

7. Conclusion

cascade of aftershock induce aftershock diffusion.

- correspondence between ETAS and CTRW

- different regimes of diffusion

- seismic diffusion occur and should be observed only when $p < 1$ and $t < t^*$

- No anomalous stress diffusion is needed.